Published by Institute of Physics Publishing for SISSA/ISAS
Received: *January 17, 2003*
Revised: *March 7, 2003*
Accepted: *March 18, 2003*

# Deconstruction, lattice supersymmetry, anomalies and branes

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Abstract: We study the realization of anomalous Ward identities in deconstructed (latticized) supersymmetric theories. In a deconstructed four-dimensional theory with $N = 2$ supersymmetry, we show that the chiral symmetries only appear in the infrared and that the anomaly is reproduced in the usual framework of lattice perturbation theory with Wilson fermions. We then realize the theory on the world-volume of fractional D-branes on an orbifold. In this brane realization, we show how deconstructed theory anomalies can be computed via classical supergravity. Our methods and observations are more generally applicable to deconstructed/latticized supersymmetric theories in various dimensions.

Keywords: Field Theories in Lower Dimensions, Lattice Gauge Field Theories, Brane Dynamics in Gauge Theories, Anomalies in Field and String Theories.




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## 1. Introduction and summary

Deconstruction was originally proposed as a gauge-invariant regularization and ultraviolet completion of five- and six-dimensional gauge theories in terms of a four-dimensional theory with a particular product gauge group, a "moose" theory [1]. (Theories with a product of $N$ gauge groups and matter fields linking the groups were considered earlier in [2] and [3], albeit with a very different motivation and interpretation.)

These "moose," or "quiver," product-group theories appear naturally in string theory as world-volume theories on branes at orbifold singularities [4]. The relation of deconstruction to string theory has led to some interesting proposals. One is the idea that a four dimensional "moose" theory may provide, in an appropriate limit, a definition of the ill-understood $(0,2)$ supersymmetric six-dimensional theory [5].

Another proposal is based on the observation that the brane realization of deconstruction can be used to give a definition of three- and four-dimensional supersymmetric gauge theories in terms of one- or zero-dimensional product-group theories. These one-(zero-) dimensional theories can provide a spatial [6] (or euclidean [7]) lattice formulation of supersymmetric theories; see also [8]. The lattice theory explicitly preserves part of the supersymmetry of the continuum theory. The explicit supersymmetry, along with other symmetries of the deconstructed theory forbids relevant operators that break the continuum supersymmetry, and ensures that the enhanced continuum supersymmetry is achieved without fine tuning [6].

This development leads to the hope that deconstruction may ultimately be useful to study strong-coupling supersymmetric gauge dynamics in three or four dimensions via numerical simulations; for a review of the progress in this direction, see [9]. This is important, because many interesting aspects of supersymmetric gauge dynamics, relevant for various applications, are not accessible by the tools — holomorphy and symmetries — normally used to obtain exact results.

An important tool in the study of nonperturbative gauge dynamics is the understanding of chiral symmetries and anomalies (both gauge and 't Hooft). In this paper, we investigate the chiral symmetries and their anomalies in the deconstructed (lattice) formulation of supersymmetric theories. We note that some related issues — the parity anomaly and the Chern-Simons term — in the deconstruction of five-dimensional theories have been addressed before [10, 11]; see also [12]. Here we study the deconstruction of four-dimensional theories, with continuous global chiral symmetries with gauge or 't Hooft anomalies. The deconstructed supersymmetric theories we study are vector-like. Applying deconstruction to chiral gauge theories can be done in principle, but is expected to face the usual problems of putting chiral fermions on the lattice. We hope to return to this issue in the near future.

We consider a simple example: the four-dimensional $N = 2$ supersymmetric pure Yang-Mills theory on a one-dimensional spatial lattice, i.e. its deconstruction in terms of a three-dimensional product-group theory. This setup is relatively simple, allowing us to look at chiral symmetries and anomalies from different points of view, while keeping the expressions reasonably tractable; yet, it contains features generic enough to be shared by other deconstructed/lattice supersymmetric theories (e.g. 4d $N = 4$, or 3d $N = 4$, SYM on a spatial lattice [6]; the deconstruction of higher-dimensional supersymmetric theories, etc.). In fact, all our findings can be generalized to these examples.

Below, we give a summary of our results and point to the relevant sections of the paper:

1. We find that the chiral symmetry of the theory is not explicit at finite lattice spacing and is broken by a Wilson term, with coefficient fixed by the (super-)symmetries of the deconstructed theory. Thus the chiral symmetry appears only as an accidental symmetry in the continuum limit. The lattice action is worked out in section 2.1, and the equivalence to the standard Wilson fermion formulation is shown in section 2.2.

2. The anomaly of the chiral $\mathrm{U}(1)_R$ symmetry can be understood using lattice perturbation theory, as in the classic work of Karsten and Smit [13]. A point worth noticing is that the large value of the Wilson term coefficient precludes the interpretation of the anomaly as due to heavy doublers acting as 4d Pauli-Villars regulators in the continuum limit, yet the correct anomalous Ward identity results. This is discussed in section 2.2.

3. A "modern" view on the chiral anomaly in the deconstructed theory results from considering its fractional brane realization. This brane realization is explained in sections 3.1, 3.2. In section 3.1, we also review how the $\mathrm{U}(1)_R$ anomaly in the continuum Seiberg-Witten theory is understood from the corresponding classical supergravity.



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4. We show, in section 3.3, how one can compute the anomaly in the lattice version of the theory using fractional branes. We also find the expression for the running coupling and discuss the continuum limit. In section 3.4, we give a geometric interpretation of the R-symmetry violating Wilson term.

To summarize, we have shown that even though the chiral symmetries are explicitly broken in the deconstructed/latticized supersymmetric theories, the correct anomalous Ward identities are obtained in the continuum limit. We show this from two different points of view: lattice perturbation theory and the brane realization of deconstruction. Our observations are more generally valid than the model we considered: for example, fractional branes can be used to study the gauge coupling running and the Chern-Simons coefficient in deconstructed $N = 1$ 5d theories.

We do not yet have a detailed understanding of the anomaly in the deconstructed theory as charge nonconservation in a nontrivial gauge field background. This question is currently under investigation; it requires a better understanding of the map between nonperturbative effects in the deconstructed and continuum theories.

## 2. Global chiral symmetries and anomalies in the deconstructed Seiberg-Witten theory

In this section, we study the global chiral symmetries of the classical action of the deconstructed $N = 2$ 4d Yang-Mills theory. We show that the global chiral $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$ symmetry of the continuum theory is explicitly broken by the lattice action. The anomaly of the $\mathrm{U}(1)_R$ symmetry is reproduced, in the continuum limit, by lattice perturbation theory, similar to [13].

### 2.1 The deconstructed theory action and symmetries

We begin by writing down the action of the deconstructed theory, starting from the superspace expression. The theory whose lattice action we want to explicitly study is the deconstruction of a 4d $N = 2$ Yang-Mills theory with gauge group $\mathrm{SU}(k)$. Its deconstruction is the dimensional reduction, from 4d to 3d, of the theory considered in [14]. It is convenient to retain four-dimensional superfield notation throughout, while simply "forgetting" the dependence on the coordinate $x^3$.

The deconstructed theory is a $\mathrm{SU}(k)^N$ quiver gauge theory[1] with $N = 2$ 3d supersymmetry (four real supercharges). The gauge fields are in vector supermultiplets $V_i$, while the chiral matter superfields $Q_i$ transform as a fundamental under the $i$-th gauge group and antifundamental under the $i+1$-th gauge group. In other words $Q_i \to e^{\Lambda_i} Q_i e^{-\Lambda_{i+1}}$, $e^{V_i} \to e^{-\Lambda_i^\dagger} e^{V_i} e^{-\Lambda_i}$ under gauge transformations. The $N+1$-th gauge group is identified

[1]For simplicity, we ignore the $\mathrm{U}(1)$ factors present in the lattice models of [6], as well as in the D-brane construction, which have a $\mathrm{U}(k)^N$ gauge group. We only note that the strength of the $\mathrm{U}(1)$ D-term potential might be used to stabilize the relative fluctuations of the $Q_i$ vevs (2.3), i.e. the difference between lattice spacings on neighboring links (but not of the overall zero mode; this issue is not relevant for our 3d to 4d deconstruction, so we do not address it here).

with the first, so that a closed circular "moose" is obtained. The superspace lagrangian is described by the tree-level Kähler potential:

$$K = \frac{1}{g_3^2}\sum_{i=1}^{N} \mathrm{tr}\; Q_i^\dagger\; e^{V_i}\, Q_i\, e^{-V_{i+1}}\,. \tag{2.1}$$

The $F$-term part of the tree-level lagrangian[2] arises from the gauge kinetic terms:

$$W = \frac{1}{2g_3^2}\sum_{i=1}^{N} \mathrm{tr} W^\alpha(V_i) W_\alpha(V_i)\,. \tag{2.2}$$

The superpotential is forbidden by the $\mathrm{U}(1)_R$ symmetry of the theory, part of the $\mathrm{U}(1)^N \times \mathrm{U}(1)_R \times Z_N$ global symmetry of the model (here $\mathrm{U}(1)_i$ is the global symmetry under which the $Q_i$ have unit charge and $Z_N$ is the "rotational" symmetry of the moose; the theory also preserves 3d parity). We keep the canonical dimensions of all fields appropriate to 4d, so $g_3^2$ has dimension of mass. We use generators in the fundamental, so that $\mathrm{tr}\; T^a T^b = \delta^{ab}/2$ and $[T^a, T^b] = i f^{abc} T^c$. Further, we also need $g^{abc} = \mathrm{tr}(T^a T^b T^c) = \frac{1}{4} i f^{abc} + \frac{1}{2} d^{abc}$, with $d^{abc} = \mathrm{tr}\; T^a\{T^b, T^c\}$.

Below, we work in Wess-Zumino gauge. We let $m = 0,1,2$ and $\mu = 0,1,2,3$. The $A_3$ component of the 4d vector boson is the real adjoint scalar in the 3d $N = 2$ vector multiplet. Even though in 3d we can use the charge conjugation matrix $\sigma^3_{\alpha\dot\alpha}$ to convert dotted into undotted indices and write the entire lagrangian in terms of one type of indices, we will keep the dotted and undotted spinor indices as this simplifies the consideration of the 4d limit.

As was shown in [14] by considering the mass spectrum, the theory (2.1), (2.2) describes, along the Higgs branch $\langle Q_i\rangle = v$ and in the large-$N$ limit, a supersymmetric field theory with gauge group $\mathrm{SU}(k)$ in one higher dimension and with twice the amount of supersymmetry, in our case — an $N = 2$ 4d supersymmetric theory.

Our aim here is to study the interactions and the symmetries of the theory. To describe the relevant (i.e. the ones charged under $\mathrm{SU}(k)^N$, keeping the vevs frozen, see footnote 1) fluctuations of the link fields $Q_i$, we define:

$$Q_i = v(1 + q_i^a T^a)\,. \tag{2.3}$$

It is straightforward to isolate the fermion bilinear terms corresponding to (2.1) and (2.2):

$$\begin{aligned} L_{\text{ferm.}} = \frac{|v|^2}{g_3^2}\sum_{j=1}^{N}\Big[ & -\frac{i}{2}\bar\psi^a_j\bar\sigma^m\partial_m\psi^a_j + \frac{i}{2\sqrt{2}}\,\psi^a_j\left(\lambda^a_j - \lambda^a_{j+1}\right) - \frac{i}{2\sqrt{2}}\,\bar\psi^a_j\left(\bar\lambda^a_j - \bar\lambda^a_{j+1}\right) - \\ & - \frac{1}{2}\, g^{abc}\;\bar\psi^a_j\bar\sigma^\mu\psi^b_j\; A^{j+1,c}_\mu + \frac{1}{2}\, g^{abc}\;\bar\psi^a_j\bar\sigma^\mu\psi^c_j\; A^{j,b}_\mu + \\ & + \frac{i}{\sqrt{2}}\, g^{abc}\Big(q^b_j\;\bar\lambda^c_{j+1}\bar\psi^a_j - q^{*,a}_j\;\lambda^c_{j+1}\psi^b_j\Big) - \frac{i}{\sqrt{2}}\, g^{abc}\Big(q^c_j\;\bar\lambda^b_j\bar\psi^a_j - q^{*,a}_j\;\lambda^b_j\psi^c_j\Big)\Big] + \\ + \frac{1}{g_3^2}\sum_{j=1}^{N}\Big[ & -i\bar\lambda^a_j\bar\sigma^m\partial_m\lambda^a_j + \frac{i}{2} f^{abc}\bar\lambda^a_j\bar\sigma^\mu\lambda^c_j A^{j,b}_\mu\Big]\,. \end{aligned} \tag{2.4}$$

[2] In the superrenormalizable $N = 2$ 3d supersymmetric theory (2.1), (2.2), there are no divergent corrections to the tree level lagrangian. Finite loop effects are discussed in section 3.3.



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The expression (2.4) will be useful in section 2.2, when comparing to the Wilson fermion action. However, we will first consider the fermion lagrangian (2.4) in momentum space along the latticized direction. Diagonalizing the kinetic terms helps to reveal the 4d nature of the theory and the enhanced supersymmetry in the continuum limit, as well as to elucidate the finite-$N$ symmetries of the fermion interactions.

To write down the Fourier space action, we will proceed in two steps, skipping the intermediate results, but explicitly giving the relevant field redefinitions. First, we perform the following Fourier transformations in (2.4), assuming odd $N$ for definiteness:

$$\begin{aligned}
\lambda_j &= \frac{1}{\sqrt{N}} \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \omega_N^{jk-\frac{k}{2}} \lambda_k \,, \qquad \psi_j = \frac{\sqrt{2}}{\sqrt{N}|v|} \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \omega_N^{-jk} \psi_k \,, \\
A_\mu^{k,a} &= \frac{1}{\sqrt{N}} \sum_{j=1}^{N} A_\mu^{j,a} \omega_N^{jk-\frac{k}{2}} \,, \qquad q_k^a = \frac{|v|}{\sqrt{2N}} \sum_{j=1}^{N} q_j^a \omega_N^{jk} \,, \\
q_k^{*a} &= \frac{|v|}{\sqrt{2N}} \sum_{j=1}^{N} q_j^{*a}\, \omega_N^{-jk} = (q_k^a)^* \,,
\end{aligned} \tag{2.5}$$

where $\omega_N = e^{\frac{2\pi i}{N}}$, and the complex conjugate transformation for the dotted fields. Note also the reality condition $A^{-k} = (A^k)^*$. It is easy to verify that terms proportional to $d^{abc}$ vanish in the continuum limit. We will thus (for simplicity only) further ignore them even for finite $N$, concentrating, e.g., on the $SP(n)$ case. Finally, note that the lagrangian is invariant under 3d parity, with $\psi(-x_2) = i\sigma^3\bar{\sigma}^2\psi(x)$, $\lambda(-x_2) = -i\sigma^3\bar{\sigma}^2\lambda(x_2)$, $A_{2(3)}(-x_2) = -A_{2(3)}(x_2)$, and $A_{0(1)}(-x_2) = A_{0(1)}(x_2)$. Invariance of the mass term requires opposite intrinsic parities of the gauginos and bifundamental fermions.

The second step in revealing the four-dimensional Lorentz invariant nature of the continuum limit spectrum is to perform yet another field redefinition:

$$\begin{aligned}
\psi_{k\alpha} &= \frac{1}{\sqrt{2}} \left( \sigma^3_{\alpha\dot{\alpha}} \bar{\chi}_k^{\dot{\alpha}} + i\eta_{k\alpha} \right), \qquad \bar{\psi}_k^{\dot{\alpha}} = \frac{1}{\sqrt{2}} \left( -\bar{\sigma}^{3\dot{\alpha}\alpha} \chi_{k\alpha} - i\bar{\eta}_k^{\dot{\alpha}} \right), \\
\lambda_{k\alpha} &= \frac{1}{\sqrt{2}} \left( \chi_{k\alpha} + i\sigma^3_{\alpha\dot{\alpha}} \bar{\eta}_k^{\dot{\alpha}} \right), \qquad \bar{\lambda}_k^{\dot{\alpha}} = \frac{1}{\sqrt{2}} \left( \bar{\chi}_k^{\dot{\alpha}} + i\bar{\sigma}^{3\dot{\alpha}\alpha} \eta_{k\alpha} \right).
\end{aligned} \tag{2.6}$$

and write the final result for the fermion bilinear lagrangian (2.4) in Fourier space as:

$$\begin{aligned}
L_{\text{ferm.}} = \frac{1}{g_3^2} \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \Big[ &- i \left( \bar{\eta}_k^a \bar{\sigma}^m \partial_m \eta_k^a + \bar{\chi}_{-k}^a \bar{\sigma}^m \partial_m \chi_{-k}^a \right) + \\
&+ |v| \sin\frac{\pi k}{N} \left( \bar{\eta}_k^a \bar{\sigma}^3 \eta_k^a + \bar{\chi}_{-k}^a \bar{\sigma}^3 \chi_{-k}^a \right) \Big] + \\
+ \frac{1}{\sqrt{N} g_3^2} \sum_{k,k'=-\frac{N-1}{2}}^{\frac{N-1}{2}} \Big[ &-\frac{i}{2} f^{abc} B_\mu^{c,k,k'} \left( \bar{\eta}_k^a \bar{\sigma}^\mu \eta_{k'}^b + \bar{\chi}_{-k}^a \bar{\sigma}^\mu \chi_{-k'}^b \right) + \\
&+ \frac{1}{2} f^{abc} \left( \eta_k^a \chi_{k'}^b \phi_{k,k'}^{*c} + \bar{\eta}_k^a \bar{\chi}_{k'}^b \phi_{k,k'}^c \right) \Big] .
\end{aligned} \tag{2.7}$$

In the continuum limit ($|v| \to \infty$) all modes except those with $k \ll N$ decouple and the lagrangian (2.7) becomes that of the 4d $N = 2$ supersymmetric Yang-Mills theory. We have defined the Fourier components (along the latticized direction) of the 4d gauge field $B_\mu$ and the 4d complex adjoint scalar $\phi$ as follows:

$$
\begin{aligned}
B_m^{c,k,k'} &= A_m^{c,k-k'} \cos^2\left(\frac{\pi(k-k')}{2N}\right) , \\
B_3^{c,k,k'} &= -\frac{i}{\sqrt{2}}\left(\cos\frac{\pi k}{N}\, q^{*,c}_{k'-k} - \cos\frac{\pi k'}{N}\, q^{c}_{k-k'}\right) , \\
\phi^c_{k,k'} &= A_3^{c,k-k'} \cos^2\left(\frac{\pi(k-k')}{2N}\right) - \frac{i}{\sqrt{2}}\left(\cos\frac{\pi k}{N}\, q^{*c}_{k'-k} + \cos\frac{\pi k'}{N}\, q^{c}_{k-k'}\right) .
\end{aligned}
\tag{2.8}
$$

Note that the third component of the 4d gauge field, $B_3$, is the imaginary part of the link fields' fluctuations, $q_i$ of (2.3), while the scalar in the 3d vector multiplet, $A_3$, combines with the real part of the link fields to form the complex 4d adjoint scalar $\phi$. Some further comments are in order:

1. The fields $\eta_k$, $\eta_{-k}$ and $\chi_{-k}, \chi_k$ in (2.7) are the left and right moving components of the two 4d adjoint fermions in the continuum limit, respectively. The free part of the fermion action respects an $\mathrm{U}(2)^N$ symmetry, with $(\eta^a_k, \chi^a_{-k})^T$ transforming as a doublet of $\mathrm{SU}(2)_k$ with unit charge under the corresponding $\mathrm{U}(1)_k$.

2. The fermion bilinear lagrangian $L_{\mathrm{ferm.}}$ preserves, even for finite $N$, a diagonal $\mathrm{SU}(2) \times \mathrm{U}(1) \subset \mathrm{U}(2)^N$ symmetry, with $k$-independent action on the fermions as described in the previous paragraph. Note that, even though the current associated with this diagonal U(1) has the correct low-energy limit to represent (the 3d part of) a 4d $\mathrm{U}(1)_R$ current, it does not obey the correct anomalous Ward identity. As will be shown below, there exists a current with the same $R$ charges for the light states, but with different $R$-charges for the heavy states, which gives the correct anomaly.

3. We should stress that the above diagonal $\mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry of the fermion bilinear terms is not a symmetry of the lagrangian and is broken by the bosonic terms — it is evident from the definition (2.8) of the 4d scalar adjoint $\phi$ that one can not even consistently define an U(1) action on $\phi$ for finite $N$ (so that the kinetic terms of $A_3$ and $q$ are invariant). We thus conclude that the classical $\mathrm{SU}(2) \times \mathrm{U}(1)$ R-symmetry of the 4d $N = 2$ theory is only exact in the continuum limit.

4. As usual in deconstruction, the mass spectrum can be read off from eq. (2.7). For $k \ll N$, it approximates the Kaluza-Klein spectrum of a four-dimensional theory compactified on a circle of radius:

$$
R = \frac{N}{\pi |v|} . \tag{2.9}
$$

The value of the lattice spacing is $a = 2/|v|$.

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The above analysis applies also to models of lattice supersymmetry with two or more latticized dimensions, but the classical action and field redefinitions turn out to be a lot more complicated and there is no point of giving them here — as we will see (see section 3.4), the D-brane picture makes the finite-$N$ violation of the chiral symmetries explicit.

### 2.2 Equivalence to Wilson fermion action and the anomaly calculation

We now go back to the $x$-space lattice lagrangian (2.4). It has already been noted that deconstruction of 5d theories gives rise to Wilson terms for the fermions[3] [15]; this, in fact, is a generic feature of deconstruction. Here, we show this using the deconstructed theory action obtained in section 2.1. More generally, in section 3.4, we will see that the Wilson term has a simple geometric interpretation in the brane realization of deconstruction.

Let us compare our lattice fermion action (2.4) to the one used in the work on anomalies with Wilson fermions of Karsten and Smit (KS) [13]. We start with the usual Wilson fermion action on a 4d lattice (see e.g. [13]). We convert to a 2-component notation, where the Dirac spinor $\psi$ of KS is[4]

$$\psi = \begin{pmatrix} \chi_\alpha \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix}, \tag{2.10}$$

set the bare mass to zero, take the continuum limit in $x^0, x^1$ and $x^2$, and obtain the Wilson fermion lagrangian with one latticized dimension:

$$\begin{aligned} L^{KS}_{\text{ferm.}} = \sum_j \Big\{ & -i\left[\bar{\eta}_j \bar{\sigma}^m \partial_m \eta_j + \bar{\chi}_j \bar{\sigma}^m \partial_m \chi_j\right] - \\ & - \frac{i}{2a}\left[\bar{\eta}_j \bar{\sigma}^3 (\eta_{j+1} - \eta_{j-1}) + \bar{\chi}_j \bar{\sigma}^3 (\chi_{j+1} - \chi_{j-1})\right] + \\ & + \frac{r}{2a}\left[\eta_j (\chi_{j+1} + \chi_{j-1} - 2\chi_j) + \bar{\eta}_j \left(\bar{\chi}_{j+1} + \bar{\chi}_{j-1} - 2\bar{\chi}_j\right)\right]\Big\}. \end{aligned} \tag{2.11}$$

We compare this to the free part of (2.4):

$$\begin{aligned} L^{\text{free}}_{\text{ferm.}} = \frac{1}{g_3^2} \sum_j \Big\{ & -i\left[\frac{|v|^2}{2} \bar{\psi}_j \bar{\sigma}^m \partial_m \psi_j + \bar{\lambda}_j \bar{\sigma}^m \partial_m \lambda_j\right] - \\ & - \frac{i|v|^2}{2\sqrt{2}}\left[\psi_j\left(\lambda_{j+1} - \lambda_j\right) - \bar{\psi}_j\left(\bar{\lambda}_{j+1} - \bar{\lambda}_j\right)\right]\Big\}. \end{aligned} \tag{2.12}$$

Recall that the lattice spacing is $a = 2/|v|$. The field redefinitions relating the deconstructed theory lagrangian (2.12) to the Wilson fermion lagrangian (2.11) are:[5]

$$\begin{aligned} \frac{|v|}{g_3\sqrt{2}} \psi_{j\alpha} &= \frac{1}{\sqrt{2}}\left(\sigma^3_{\alpha\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}_j - i\eta_{j\alpha}\right), \\ -\frac{1}{g_3} \lambda_{j\alpha} &= \frac{1}{\sqrt{2}}\left(\chi_{j\alpha} - i\sigma^3_{\alpha\dot{\alpha}} \bar{\eta}^{\dot{\alpha}}_j\right). \end{aligned} \tag{2.13}$$

[3] One of us (E.P.) thanks Chris Hill for discussions.

[4] $\psi$, $\eta$ and $\chi$ here should not be confused with the spinors of the previous subsection.

[5] Note that these are related to (2.6) by $k \to j$ and $\eta \to -\eta$; the sign is important in the Wilson mass term.

The matching of the two lagrangians shows that $r = 1$ in the deconstructed theory. This value of the Wilson term is imposed by the explicit supersymmetry and gauge invariance of the deconstructed theory.

In the four-component basis of (2.10), the axial U(1) current is associated with $\psi \to e^{-i\varphi\gamma_5}\psi$ with $\gamma_5 = \mathrm{diag}(-1, 1)$. Making this local we have in the 2-component notation:

$$\eta_j \to e^{i\varphi_j}\eta_j\,, \qquad \chi_j \to e^{i\varphi_j}\chi_j\,. \tag{2.14}$$

This axial transformation in our deconstructed theory is the anomalous $\mathrm{U}(1)_R$ transformation of the continuum 4d theory, combined with a diagonal generator of the anomaly free $\mathrm{SU}(2)_R$.

The variation of the lattice action (2.11), or, equivalently, (2.4) written in terms of the new variables (2.13), under the axial U(1) is:[6]

$$\begin{aligned}\frac{\delta L^{KS}_{\text{ferm.}}}{\delta\varphi_j} &= -\partial_m\left[\bar\eta_j\bar\sigma^m\eta_j + \bar\chi_j\bar\sigma^m\chi_j\right] - \\ &\quad -\frac{1}{2a}\left[\left(\bar\eta_j\bar\sigma^3\eta_{j+1} + \bar\chi_j\bar\sigma^3\chi_{j+1} + \text{h.c.}\right) - \left(\bar\eta_{j-1}\bar\sigma^3\eta_j + \bar\chi_{j-1}\bar\sigma^3\chi_j + \text{h.c.}\right)\right] + \\ &\quad +\frac{r}{2a}\left[i\eta_j(\chi_{j+1} + \chi_{j-1} - 2\chi_j) + i\chi_j(\eta_{j+1} + \eta_{j-1} - 2\eta_j) + \text{h.c.}\ \right].\end{aligned} \tag{2.15}$$

In the continuum limit, the second term in brackets becomes:

$$-\partial_3\left[\bar\eta\bar\sigma^3\eta + \bar\chi\bar\sigma^3\chi\right]. \tag{2.16}$$

Thus the anomalous Ward identity is given by:

$$\langle\partial_\mu\left[\bar\eta\bar\sigma^\mu\eta + \bar\chi\bar\sigma^\mu\chi\right]\rangle_{x^3_{(j)}} = \lim_{a\to 0}\frac{r}{2a}\langle[i\eta_j(\chi_{j+1} + \chi_{j-1} - 2\chi_j) + i\chi_j(\eta_{j+1} + \eta_{j-1} - 2\eta_j) + \text{h.c.}\ ]\rangle. \tag{2.17}$$

The right-hand side is nothing but the "axial symmetry breaker $D_0^A$" of KS.[7] To obtain this result, note that $M = r/a$ (in the notation of KS, eq. (5.7) of [13]) since we only have a Wilson term in one dimension and the bare mass is zero. In addition, we make use of the identity $-\bar\psi_j\gamma_5\psi_{j'} = \eta_j\chi_{j'} - \bar\chi_j\bar\eta_{j'}$ for the Dirac spinor (2.10) taken at different points $j, j'$ along the latticized dimension.

Having thus established the equivalence of the deconstructed theory action to the Wilson fermion action and having matched the corresponding anomalous Ward identities, we are assured that the remainder of the weak coupling axial anomaly calculation goes through according to KS. The latticization of one dimension regulates the linear divergence usually associated with the weak coupling anomaly; furthermore the regulator respects the vector current Ward identity and Bose symmetry; hence we are assured that the axial anomaly emerges just as if we had latticized all 4d, as KS have done.

One difference with the work of KS is worth pointing out. In a nonsupersymmetric lattice theory, one has the freedom to vary the Wilson term coefficient $r$ at will. For $r = 0$, the doublers have a continuum 4d interpretation as massless fermions of opposite

[6] Of course, $\delta L^{\text{free}}_{\text{ferm.}}/\delta\varphi_j$ of eq. (2.4) is obtained from this by taking $r = 1$.

[7] Up to an overall normalization. Also note that in the notation of KS, $\lambda_0 = 1$ corresponds to the flavor-diagonal current.



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axial charge which cancel the anomaly of the physical fermions. For $0 < r < 1/\sqrt{2}$, the doublers still have a 4d continuum interpretation as fermions of opposite axial charge, but with chiral symmetry violating masses. This follows from the Wilson fermion dispersion relation, which can be read off eq. (2.11), or, more easily, from the geometric D-brane picture, see eqs. (3.26), (3.27):

$$a^2\omega^2(k) = \sin^2\frac{2\pi k}{N} + 4r^2\ \sin^4\frac{\pi k}{N}\,. \tag{2.18}$$

(Here $\omega$ is the energy and for simplicity we have set the continuum momenta $p^1, p^2$ to zero.) Thus (for $0 < r < 1/\sqrt{2}$) the effect of the doublers can be "physically" described as that of Pauli-Villars regulators, whose axial symmetry violating mass terms give rise to the anomaly. In our case, $r = 1$, it is easy to see that there is no continuum interpretation of the would-be "doublers" as 4d massive Dirac fermions; nevertheless, as KS showed, the calculation of the anomaly is valid for all values of $r$.

## 3. Fractional branes and the chiral anomaly via classical supergravity

In this section, we show that the lattice perturbation theory result for the $\mathrm{U}(1)_R$ anomaly can be obtained via a classical supergravity calculation in the fractional brane realization of the lattice supersymmetric models. We should stress that the brane construction is very general and underlies all proposed realizations of lattice supersymmetry by deconstruction. Thus, stringy methods should be of use in elucidating the symmetries of the lattice theories and their anomalies.

In particular, if the brane constructions are such that string perturbation theory is valid, one should recover the lattice calculation of the anomaly in the $\alpha' \to 0$ limit of string perturbation theory. Moreover, if the target theory has a large amount of supersymmetry ($\geq 8$ supercharges), string perturbation theory reduces to classical supergravity (see discussion below), which makes the calculation of the gauge anomaly (and, in these theories, the closely related running coupling) even more straightforward. Even though we apply supergravity methods to study the anomaly in the theory considered in section 2, it is straightforward to generalize to all cases listed in foonote 11 (section 3.2); the parity anomaly in the deconstructed 5d supersymmetric case can be also obtained in exact parallel of our considerations here (by essentially replacing the Coulomb-like potentials of section 3.3 with logarithmic ones).

For completeness, we begin in section 3.1 by reviewing the fractional brane construction of the continuum limit 4d $N = 2$ theory and describing the calculation of the continuum theory $\mathrm{U}(1)_R$ anomaly. This section also reviews well-known results about the closed string calculation of the running coupling in supersymmetric gauge theories.

The fractional brane construction of the deconstructed $\mathrm{SU}(k)^N$ theory is given in section 3.2. Then, in section 3.3, we study the running of the gauge coupling in the deconstructed theory. We compare the running coupling calculation (section 3.3.1) with the field theory (section 3.3.2) and study the continuum limit in the fractional brane picture. The supergravity calculation of the anomaly is discussed in section 3.3.3. Finally, in section 3.4, we give a geometric picture of the R-symmetry and the symmetry-breaking Wilson term.

### 3.1 Anomalies in Seiberg-Witten theory via fractional branes

First, let us derive the non-abelian anomaly and $\beta$-function of the Seiberg-Witten theory from an appropriate orbifold. This will also be the desired result in the continuum limit of the deconstruction setup.

In order to realize a four dimensional theory with eight supercharges, we take D3 branes stretched along $X^0, \ldots, X^3$. The D3 branes are transverse to a $C^2/Z_r$ orbifold acting in the standard way on the coordinates $X^4, \ldots, X^7$. Here $r \geq 3$, but otherwise arbitrary.

This gives a quiver gauge theory [4] on the brane, whose matter content depends on the choice of representation of the Chan-Paton (CP) factors. The simplest such choice, namely the one-dimensional trivial representation of $Z_r$, gives a pure gauge theory.[8] Note that the more common choice — the regular representation — is $r$-dimensional, allowing for branes to move away from the fixed point. Here the brane is stuck at the fixed point (i.e. its world-volume theory has no Higgs branch), and is dubbed a "fractional brane." Of course one can have any multiplicity for these representations, giving non-abelian theories.

These fractional branes behave slightly differently from the usual D3 branes [16]. For example, when calculating their kinetic term, it is a different modulus than the complexified string coupling that controls their gauge coupling. This can be demonstrated by an explicit worldsheet calculation, as sketched below.

The closed string sector has $r-1$ twisted sectors, and each one of them has in the spectrum a massless complex scalar (propagating in the 6 dimensions transverse to the orbifold). We denote these moduli by $\tau_i$, where for a general orbifold $i$ runs over all the nontrivial elements of the discrete group. Here $i = 0, \ldots, r-1$, where we denote by $\tau_0$ the usual (untwisted) complexified string coupling.

Generally, to specify a particular quiver theory one chooses a representation of the CP factors of some dimension, say $m$. Here $m = 1$, while for the regular representation $m = r$ (we discuss here the general case, which will be useful later). This means that every element of the group is represented by a matrix $\gamma_i$ of dimension $m$. The gauge fields carry a (diagonal) $m \times m$ matrix $A$. For non-abelian theories we simply take some multiplicity of the above structure.

One can now proceed to calculate the gauge theory action from various worldsheet couplings. The actual worldsheet amplitude will be only a (non-zero) irrelevant factor. All that matters is various selection rules one gets from the CP part of the amplitudes.

For example, in calculating kinetic terms in the action, one has to calculate the amplitude with two open string vertex operators and one closed string modulus. The dependence on the CP factors is encoded in the trace $\mathrm{Tr}(A\gamma_i A)$. For example, the usual case — with the regular (traceless) representation for $\gamma_i$ — has $A$ proportional to the unit matrix, therefore $\mathrm{Tr}(A\gamma_i A) = 0$ unless $i = 0$. One then deduces that $\tau_0$ is the gauge coupling on the brane.

[8] Obtaining a theory with matter of the quiver form is done simply by including more representations of $Z_r$. This should not present any essential new features for our purposes here.



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For the fractional brane, on the other hand, the dimension $m = 1$ and $\gamma_i = \omega_r^i$, where $\omega_r$ is any $r$-th root of unity. If we choose for simplicity this root of unity to be 1, then all the generators of $Z_r$ are simply represented by that number. With this choice, our fractional brane couples identically to all fields $\tau_i$ ($i = 0, \ldots, r-1$). This is more conveniently summarized if we perform a discrete Fourier transform on the closed string fields:

$$\tau_m = \sum_{i=0}^{r-1} \omega_r^{im} \tau_i. \tag{3.1}$$

We see therefore that our fractional brane couples to the field $\tau_{m=0}$ only. The other possible choices of $r$-th root of unity lead to fractional branes, each of which couples to a specific Fourier mode of the closed string fields.

Having identified the holomorphic gauge coupling, we are now ready to discuss quantum effects. Suppose one wants to calculate the running coupling of a non-abelian gauge theory SU($k$) at some scale $|A|$. This can be achieved, for example, by going to a particular point of its Coulomb branch, where a VEV of magnitude $|A|$ breaks the gauge symmetry to $\mathrm{SU}(k-1) \times \mathrm{U}(1)$. The low-energy effective action (valid at scales below $|A|$) then contains a kinetic term for the abelian field:

$$\int d^2\theta\, \tau(A)\, W^\alpha W_\alpha\,, \tag{3.2}$$

and the value of the U(1) (frozen) gauge coupling is equal to the SU($k$) (running) gauge coupling, evaluated at the scale of the breaking $|A|$.

In the field theory, the abelian gauge coupling at $A$ (and below) is obtained by integrating out the massive W-boson supermultiplets which are charged under the U(1). In a brane realization the SU($k$) symmetry lives on a set of $k$ D-branes, which we presently separate to one group of $k-1$ branes, which we call the "source branes," and a single brane, called the "probe brane." The massive W-bosons are then strings stretched between the source and the probe ($s-p$ strings). We emphasize that the distinction between the source and the probe is a matter of convenience, and applies even in the $k = 2$ case.

At one loop, the result can be extracted from an annulus diagram with two massless open-string vertex operators, corresponding to U(1) gauge bosons, inserted on the boundary. In order to isolate the contributions of the $s-p$ strings alone, the boundary conditions on both boundaries are chosen differently: one of them (the one with the gauge vertex operator insertions) in correspondence to the probe brane, and the other in correspondence to the source branes. The mass of the W-bosons is proportional to the separation between the branes, so in order to decouple the open-string oscillator modes, one works in the limit of sub-stringy separations.

The supergravity analysis relies on the old observation that the annulus diagram can be calculated in the closed-string channel as well. The limit appropriate for a supergravity treatment is that of large separations compared to the string length, but supersymmetry [17] aids in making the extrapolation to the sub-stringy, field theory, regime.



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In the closed-string channel, the annulus factorizes into a sum of all possible intermediate closed string fields. In the supergravity limit only the massless fields are relevant. Each contribution is then a product of three factors:

1. The tadpole: a disc diagram with one closed-string insertion, and boundary conditions corresponding to the source branes. In the supergravity action this is summarized by a source term for the closed-string field. From the above discussion we see that only the field $\tau_{m=0}$ has a tadpole. The tadpole in our case is proportional to $k$, the total number of branes.[9] We choose to normalize the closed-string fields such that the tadpole is $\frac{b_0}{2\pi} = \frac{k}{\pi}$, where $b_0 = 2k$ is the one-loop $\beta$ function coefficient of SU$(k)$.

2. Closed-string propagator: since the twisted sector fields are allowed to vary only in two directions, this gives a logarithmic dependence on the coordinates.[10] Using complex notation for the $X^8, X^9$ plane, $z \equiv X^8 + iX^9$, one gets:
$$\Phi(z) = c + \frac{b_0}{2\pi}\log(|z|)\,, \tag{3.3}$$
where $c$ is an integration constant. $\Phi(z)$ is a NS sector scalar, which we relate to the coupling constant below.

3. The response: a disc diagram with two open-string vertex operators, and one closed-string vertex operator. This summarizes the coupling of the probe to the closed-string field. The interaction of the probe occurs at the location $z = A$, and its strength can be chosen to be $1/4\pi$ (by normalizing the open string gauge fields appropriately), therefore the coupling on the probe is:
$$\frac{1}{g^2_{\text{eff}}} = \frac{c}{4\pi} + \frac{b_0}{8\pi^2}\log(|A|)\,. \tag{3.4}$$
This indicates that the integration constant $c$ should be related to the bare coupling by $\frac{c}{4\pi} = \frac{1}{g_0^2}$.

We can use holomorphy to deduce the imaginary part of the coupling (3.4). Defining as usual $\tau(A) = \frac{\Theta(A)}{2\pi} + i\frac{8\pi^2}{g^2(A)}$, one gets,
$$\tau(A) = \tau_0(A) + i\, b_0 \log A\,. \tag{3.5}$$
In the fractional brane realization of the $N = 2$ theory, the anomalous $R$-symmetry is the SO(2) rotation in $X^8, X^9$ directions, i.e. $z \to e^{i\alpha}z$ implies $A \to e^{i\alpha}A$. (The SU$(2)_R$ comes from action on the orbifold directions, or more generally on the complex structures of $K3$.) Therefore the result (3.4) encodes both the running coupling and the R-symmetry violation.

[9] And not to $k-1$. This comes from the contribution of CP factors to the factorized closed-channel amplitude: the source's coupling to the tadpole has a factor of Tr $1 = k$.

[10] In our conventions for the normalization of the closed-string field kinetic terms, the (2 dimensional) propagator satisfies $\nabla^2 G_2(x,y) = 2\pi\delta(x)\delta(y)$.

### 3.2 The deconstruction setup with fractional branes

We now consider the deconstruction of the same theory. Generally, there are two ways to view deconstruction as a string theory setup:

1. As an orbifold of a Hanany-Witten setup [18], where D-branes are stretched between NS fivebranes. In the continuum limit (moving asymptotically on the Higgs branch) the D-branes combine and move away from the orbifold in one direction along the NS fivebranes. We will not need this setup below, so we will not provide any details here.

2. As a setup of fractional branes on an orbifold. This is the description we adopt as it allows for a perturbative string theory treatment [19].

In our case — the deconstruction of 4d Seiberg-Witten theory — we start with D2 branes[11] stretched along the $X^0, X^1, X^2$ directions. The orbifold group is chosen to be $Z_r \times Z_N$ (we take $r, N$ to be relatively prime), which acts as follows:

- The generator of $Z_N$ acts on the directions $X^3 + iX^8, X^4 + iX^5$ in the standard way. In the deconstruction setup one needs to move into the Higgs branch, so our branes will need to be in the regular representation of $Z_N$.

- The generator of $Z_r$ acts on the directions $X^4 + iX^5, X^6 + iX^7$. The CP assignments for this generator are as before (section 3.1) only taken with multiplicity of $kN$, to account for all images with respect to the $Z_N$ orbifold action, and to generate a nonabelian SU$(k)$ gauge group in four dimensions.

Thus, our CP matrices are $kN$ dimensional. Let us work out the CP factors corresponding to the various bosonic fields. Denote the action of the generator $Z_N$ on the CP factors by $\gamma_N$ and take the regular representation, tr $\gamma_N = 0$:

$$\gamma_N = \mathrm{diag}\left(\omega_N, \omega_N^2, \ldots, \omega_N^N\right), \tag{3.6}$$

where every element is multiplied by a $k \times k$ unit matrix. The gauge field's ($A_m$, $m = 0, 1, 2$) CP factors are (generally complex) $kN \times kN$ matrices, which we denote by $\mathbf{A}$. They obey:

$$\mathbf{A} = \gamma_N \mathbf{A} \gamma_N^{-1}, \tag{3.7}$$

which means that the states that are not projected out by the $Z_N$ orbifold have block diagonal CP factors:

$$\mathbf{A} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda_2 & \ldots & 0 & 0 \\ 0 & 0 & \ldots \lambda_i \ldots & 0 & 0 \\ 0 & 0 & \ldots & \lambda_{N-1} & 0 \\ 0 & 0 & \ldots & 0 & \lambda_N \end{pmatrix}, \tag{3.8}$$

where the $\lambda_i$ are the $k \times k$ CP matrices of the unbroken U$(k)^N$ gauge groups.

[11] The spatial lattice constructions of [6] can be obtained by similar (fractional) brane constructions: the 3d $N = 4$ theory is obtained upon replacing our fractional D2 branes with fractional D0 branes, and the $C^2/Z_r \times Z_N$ orbifold by a $C^3/Z_r \times Z_{N_1} \times Z_{N_2}$ orbifold; the 4d $N = 4$ theory, on the other hand, involves only regular D0 branes on a $C^3/Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ orbifold.

Similarly, we take as the generator of the $Z_r$ action $\gamma_r = \mathrm{diag}(\omega_r, \ldots, \omega_r)$, with $\omega_r$ the one-dimensional representation of $Z_r$. It is evident that all gauge boson states with CP factors (3.8) survive the $Z_r$ orbifold projection.

We now move on to describing the scalar fields. One scalar field that survives the combined $Z_N \times Z_r$ action corresponds to motion of the D2 branes in $X^9$ (the scalar of the 3d $N = 2$ vector multiplet). This scalar is described by the same CP factor as the gauge field (3.8).

The other set of surviving scalars correspond to moving in the $X^3 + iX^8$ directions, and make up the link fields (bifundamentals) of the deconstruction setup. The link fields CP factors $\mathbf{\Phi}$ obey:

$$\mathbf{\Phi} = \omega_N \gamma_N \ \mathbf{\Phi} \gamma_N^{-1} \,, \tag{3.9}$$

which is solved by:

$$\mathbf{\Phi} = \begin{pmatrix} 0 & \phi_{1,2} & 0 & \ldots & 0 & 0 \\ 0 & 0 & \phi_{2,3} & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots \phi_{i,i+1} \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & \phi_{N-1,N} \\ \phi_{N,1} & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \tag{3.10}$$

where $\phi_{i,i+1}$ are the $k \times k$ CP matrices of the $N$ link fields. In fact, the CP factors of the link fields above correspond to our fields $\phi_{i,i+1} \to Q_i$ of eq. (2.1) in the coordinate representation.

The moduli space of the $kN$ D2 branes on the orbifold is three-dimensional — branes are allowed to move in groups of $k$, in a $Z_N$ symmetric way in $X^3 + iX^8$ (Higgs branch), as well as in $X^9$ (Coulomb branch). The deconstructed theory is obtained upon moving them a distance $v$ from the origin in $X^3 + iX^8$, i.e. along the Higgs branch, as explicitly described in section 2.1.

To proceed with the analysis of gauge coupling running and the anomaly, we need the couplings of the world-volume open string massless modes to the closed string fields. This is a particular example of compactification of type-IIA string theory on a Calabi-Yau manifold, and the closed string spectrum is well-known (see e.g. [4]); we review the orbifold analysis for the sake of completeness.

A priori there are $Nr$ twisted sectors of closed strings to consider. However, to be sourced by our D-brane setup, the closed string fields need to be a particular combination of the twisted sectors with respect to the $Z_r$ orbifold, as described in the previous section.

Therefore, for our purposes there are only $N$ non-trivial sectors to consider. These are $N - 1$ twisted sectors, and one untwisted sector, with respect to the $Z_N$ action. To see the matter content in these sectors it is useful to ignore the $Z_N$ orbifold at first. In this case the twisted sector fields fill out six dimensional $(1, 1)$ vector multiplets.[12] The bosonic fields in each one of those multiplets consist of a real vector (from the R-R sector) and 4 scalars (from the NS-NS sector).

[12]The untwisted sector contains also a 6 dimensional gravity multiplet, which decouples from our analysis.

Subsequently imposing the $Z_N$ projection yields a 3-dimensional $N = 4$ vector multiplet for each of the $N$ sectors relevant for us.[13] The bosonic closed string fields surviving the complete orbifold projection are then complex scalar fields $\Psi_i$ (from the NS-NS sector), a vector field $V_i$ (from the R-R sector) and a real 0-form $S_i$ (from the R-R sector), where $i = 0, \ldots, N-1$. As usual, the $i = 0$ case refers to the untwisted sector with respect to $Z_N$, and as mentioned above all the fields are assumed to be the correct linear combination of twisted sectors with respect to the $Z_r$ action (the $m = 0$ Fourier mode, in the notations of the previous section, eq. (3.25)).

The charges of the D-branes with respect to these fields can be calculated by a disc diagram with no external states and a closed string twisted state vertex operator insertion. Since we are interested in the configuration of separated fractional branes, we calculate the charges of each fractional brane separately.

The only new element here is the factor that comes from the CP matrices of the $Z_N$ action. If we are interested in the $l$-th fractional brane, then the generator of the $Z_N$ action is represented by the CP factor $w_N^l \, 1_{k\times k}$. Therefore the coupling to any of the $q$-th twisted sector (with respect to $Z_N$) closed string fields is proportional[14] to $\frac{k}{N}\, w_N^{ql}$.

The four dimensional couplings, in the continuum limit, are T-dual to the closed string fields in our brane setup. It is clear then that we are interested only in the untwisted sector fields $q = 0$ (with respect to $Z_N$).[15] We denote these fields by $\Psi$ (a complex scalar field), $V$ (a vector field), and $S$ (a 0-form). We choose the normalizations of the closed string fields such that the coupling of the individual fractional branes to these fields is $\frac{b_0}{N}$, with no phase factors. This follows from the standard relation between different brane tensions ($T_2 = 2\pi T_3$, for $\alpha' = 1$).

### 3.3 The running coupling and the anomaly

#### 3.3.1 Supergravity calculation

Let us elaborate on the equations of motion for the closed string fields sourced by the D2 brane configuration (shown in figure 1). The sources for the tadpole are localized in the $X^3, X^8, X^9$ space at the locations:

$$\begin{aligned} X_k^3 &= v \sin \frac{2\pi k}{N}\,, \\ X_k^8 &= v \cos \frac{2\pi k}{N}\,, \\ X_k^9 &= 0\,, \end{aligned} \tag{3.11}$$

i.e. the branes are distributed with equal spacing on a circle of circumference $2\pi v$ in the $X^3$–$X^8$ plane.

[13]The untwisted sector with respect to $Z_N$ contains an additional hypermultiplet, which is irrelevant for our analysis.

[14]Note that we choose the trace over the CP matrices to be normalized such that $\mathrm{Tr}(1_{N\times N}) = 1$. In particular, in these conventions the charge of a brane in the regular representation (which has all $N$ images) is chosen to be 1.

[15]These fields are the parameters of the 3 dimensional action.

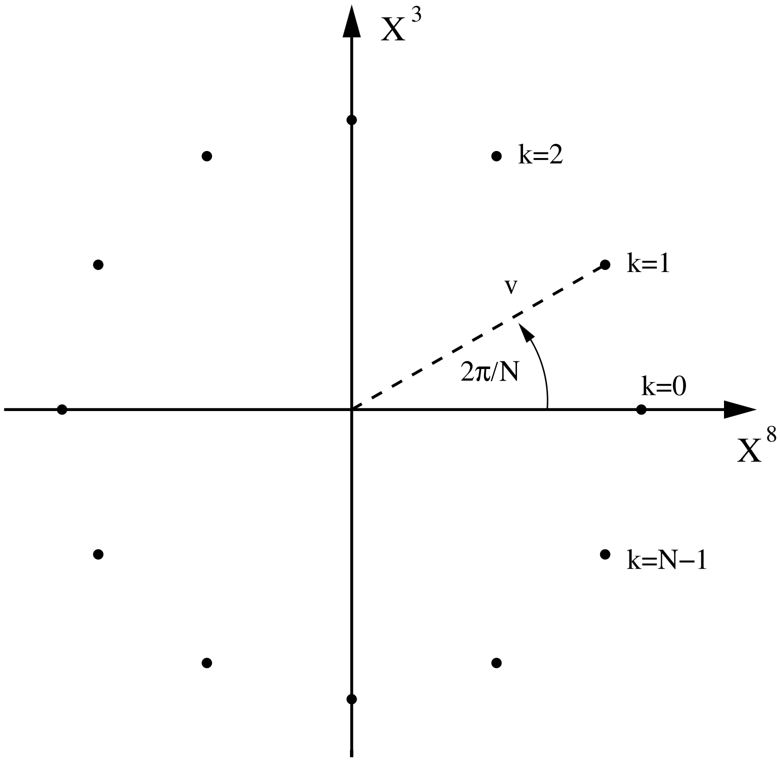


**Figure 1:** D-brane sources of the $Z_r$-twisted fields $\Psi$, (3.12), and $U$, (3.22), in a $Z_N$-symmetric distribution along a circle of radius $v$ in the $X^3$-$X^8$ plane, at $X^9 = 0$. The Coulomb branch direction $X^9$ is perpendicular to the drawing. The branes are labeled by values of $k$, as defined in (3.11).

Each one of the source terms is of strength $b_0/N$ in the normalization of the last section. The NS-NS sector scalar field $\Psi$ couples to the branes as electric sources,[16] i.e.:

$$\nabla^2\Psi = \frac{b_0}{N}\sum_{k=0}^{N-1}\delta^{(3)}\left(X - X^k\right), \tag{3.12}$$

which yields for the value of $\Psi$ at arbitrary $X^3, X^8, X^9$:

$$\Psi\left(X^3, X^8, X^9\right) = C - \frac{b_0}{4\pi N}\sum_{k=0}^{N-1}\frac{1}{\sqrt{(X^3 - X^3_k)^2 + (X^8 - X^8_k)^2 + (X^9)^2}}. \tag{3.13}$$

where $C$ is a constant.

In order to see a 4d behavior, we need to focus to a neighborhood close enough to the circle of D-branes, such that in the large-$N$, large-$v$ limit, a nearby section of the circle approximates a straight line. To that effect, we place the probe brane at a position close to the sources. Since the probe brane is also in the regular representation with respect to $Z_N$, this will include all images of that brane

[16]We choose to normalize coordinates such that the 3d propagator satisfies $\nabla^2 G_3 = \delta(x)\delta(y)\delta(z)$; therefore we have in comparison to the previous section $\int_{-R}^{R} dz\, G_3(x, y, z) \simeq 2\pi\, G_2(\frac{x}{R}, \frac{y}{R})$ for $x, y \ll R$.

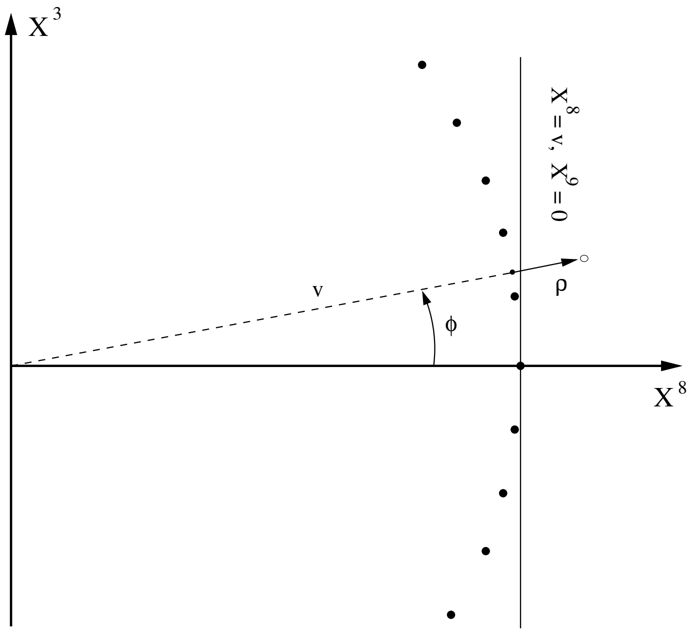


**Figure 2:** The location of a probe brane (at general $\phi$), shown here by an open circle, in relation to the arc of source branes in the neighborhood of $X^8 = v,\ X^3 = X^9 = 0$. According to (3.14), $\rho$ measures the radial displacement of the probe brane from this arc (of radius $v$), while $\phi$ gives the angular displacement from the $X^8$ axis. The $X^9$ displacement $\eta$ of the probe brane is not shown; the figure should be thought of as a projection onto the $X^9 = 0$ plane. The vertical line passing through $X_8 = v$ is the R-symmetry axis of section 3.4.

The continuum limit of the deconstructed theory is best exhibited upon introducing appropriate, see figure 2, coordinates[17] (everything here and below is made dimensionless in terms of the string length $l_s$) $\rho, \eta, \phi$ defined as:

$$\begin{aligned} X^3 &= (v+\rho)\sin\phi\,, \\ X^8 &= (v+\rho)\cos\phi\,, \\ X^9 &= \eta\,. \end{aligned} \tag{3.14}$$

The probe brane is placed[18] at constant $\rho, \eta$, and at $\phi = \frac{2\pi k'}{N}$, with $k' = 0, \ldots, N-1$ counting its images.

In contrast to the previous section, the normalization of 3-dimensional gauge fields is chosen such that the probe couples to $\Psi$ with strength $1/2$. In particular, with these conventions the 3 dimensional actions does not have an overall $1/N$ factor.

In the absence of the source, the gauge coupling on the probe brane is determined by the constant of integration, $C$ of eq. (3.13). With the source, one finds for the gauge

[17] See also discussion about coordinate choice in section 3.4

[18] This choice of $\phi$-coordinate for the position of the probe brane indicates that we do not turn on a Wilson line for the 4d gauge field even at finite $R$, since, in any case, it would vanish in the continuum $R \to \infty$ limit.

coupling on the probe:

$$\begin{aligned}
\frac{1}{g^2_{\rm probe}} &= \frac{1}{2}\sum_{k'=1}^{N} \Psi\left(\rho,\eta,\phi=\frac{2\pi k'}{N}\right) \\
&= \frac{NC}{2} - \frac{b_0}{16\pi v} \sum_{(k-k')=-\frac{N-1}{2}}^{\frac{N-1}{2}} \left[\left(1+\frac{\rho}{v}\right)\sin^2\left(\frac{\pi(k-k')}{N}\right)+\frac{\rho^2+\eta^2}{4v^2}\right]^{-\frac{1}{2}} \\
&\simeq \frac{NC}{2} - \frac{b_0 N}{16\pi^2 v}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin^2 x+\frac{\rho^2+\eta^2}{4v^2}}} \simeq \frac{NC}{2}+\frac{b_0 N}{8\pi^2 v}\log\frac{|\rho+i\eta|}{2v}\,,
\end{aligned}$$

where we took $1+\rho/v\simeq 1$, and replaced the sum[19] by an integral in the large-$N$ limit. Note that the last equality in (3.15) is only valid for $|\rho+i\eta|\ll v$; the tadpole falls off as an inverse power of $|\rho+i\eta|$ at large values of the argument.

In the next section, we will see that the probe-brane coupling, eq. (3.15), reproduces the running coupling of the continuum theory and will relate the constant $C$ to the "bare" 4d coupling. In order to continue with the interpretation of eq. (3.15), we thus remind ourselves of the deconstructed field theory calculation of the running coupling.

### 3.3.2 Field theory calculation and the continuum limit

We want to relate the tadpole $\Psi(\rho,\eta,\phi)$ of eq. (3.15) to the running of the gauge coupling in the field theory, evaluated at the scale $|A|=|\rho+i\eta|$. The gauge coupling on a probe brane placed at $\rho+i\eta$, away from the circle, will depend on the background $\Psi$, as described in section 3.2 for the continuum limit theory. For simplicity we will only focus on the $\mathrm{SU}(2)^N$ case.

The probe brane gauge coupling is the coupling of the unbroken U(1) on the Coulomb branch of the diagonal $\mathrm{SU}(2)\subset\mathrm{SU}(2)^N$ 3d gauge group — moving a brane away from the circle of radius $v$ in the $X^3, X^8$ plane is equivalent to going on the Coulomb branch. The running of the coupling is due to the heavy gauge bosons and superpartners which acquire masses on the Coulomb branch. The holomorphic gauge coupling "runs" only at one loop — more precisely, upon integrating out a (3d $N=4$) vector multiplet of mass $m$, the 3d diagonal gauge coupling $g_{3,D}$ receives a threshold correction:

$$\frac{1}{g^2_{3,D}}_{\rm low} = \frac{1}{g^2_{3,D}}_{\rm high} - \frac{1}{2\pi m}\,, \qquad (3.15)$$

where the coefficient was calculated in [21]; the minus sign indicates that the 3d theory becomes more strongly coupled in the infrared.

To calculate the gauge coupling of the U(1) at the scale $A$, we have to sum the thresholds (3.15) from all heavy states charged under the U(1), the $W^\pm$ bosons. We start with

[19] See [20].

the tree level value of the diagonal gauge coupling at the cutoff scale[20] $v$:

$$\frac{1}{g_{3,D}^2} = \frac{N}{g_3^2}\,, \tag{3.16}$$

as is usual in deconstruction. A short calculation gives for the $W_k^\pm$ boson masses along the Coulomb branch, for $|A| \ll v$:

$$\left(m_k^\pm\right)^2 = 4v^2 \sin^2\frac{\pi k}{N} + \left(\rho^2+\eta^2\right) = 4v^2\sin^2\frac{\pi k}{N} + |A|^2\,. \tag{3.17}$$

The coupling of the unbroken diagonal U(1) at the scale $|A| = |\rho + i\eta|$ is given by integrating out all $W^\pm$ since they are all heavier than $A$; recall that we are working in a normalization where $1/g_3^2$ multiplies the entire lagrangian (2.1), (2.2):

$$\begin{aligned}\frac{1}{g_{3,D}^2(|A|)} &= \frac{N}{g_3^2} - \frac{1}{2\pi}\sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \frac{1}{\sqrt{4v^2\sin^2\frac{\pi k}{N} + |A|^2}} \\ &\simeq \frac{N}{g_3^2} - \frac{N}{4\pi^2 v}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin^2 x + \frac{|A|^2}{4v^2}}}\,.\end{aligned} \tag{3.18}$$

This is, of course, a regulated version of the continuum calculation, where (in the first equality, before replacing the sum with an integral) we would have an infinite sum of Kaluza-Klein tower thresholds (3.15) with $m_k = k^2R^{-2} + |A|^2$.

The diagonal gauge coupling $g_{3,D}^2(|A|)$ of the deconstructed theory (3.18) is identified, upon dimensional reduction, with the running coupling of the continuum theory, see [14, 10]. The 4d running coupling $1/g_4^2(|A|)$ is thus obtained upon multiplication of (3.18) by $(2\pi R)^{-1}$ and replacing the sum with an integral at large $N$:

$$\frac{1}{g_4^2(|A|)} \equiv \frac{v}{N g_{3,D}^2(|A|)} = \frac{v}{g_3^2} - \frac{1}{4\pi^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{dx}{\sqrt{\sin^2 x + \frac{|A|^2}{4v^2}}} = \frac{v}{g_3^2} + \frac{1}{2\pi^2}\ln\frac{|A|}{2v}\,. \tag{3.19}$$

The coefficient in front of the logarithm is the one appropriate for the 4d $N=2$ pure SU(2) Yang-Mills theory, which is usually written as $b_0/(8\pi^2)$, with $b_0 = 4$. The continuum limit in the deconstructed theory is then achieved by taking (with $\Lambda$ fixed):

$$N\to\infty\,, \qquad v\to\infty\,, \qquad \frac{g_3^2}{v}\to 0\,, \qquad \Lambda^4 = v^4 e^{-\frac{8\pi^2 v}{g_3^2}}\,, \qquad \Lambda \ll v\,. \tag{3.20}$$

Finally, to compare with the supergravity calculation, we note that the coupling on the probe brane, eq. (3.15) with $b_0 = 4$, is identical to the coupling of the diagonal 3d gauge group (3.18). Comparing (3.18) with the tadpole (3.15) also gives a precise relation between the integration constant $C$ and the tree level coupling of the 3d theory: $C/2 = 1/g_3^2$.

[20] Hereafter, our normalization of the vev (2.3) of $Q_i$ differs from that of section 2: in the brane picture, it is conventional to have $2\pi R = N/v$, as opposed to (2.9).



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### 3.3.3 Supergravity and the $\mathrm{U}(1)_R$ anomaly

It is clear from the supersymmetry of the problem that supergravity should also yield the correct expression for the $\mathrm{U}(1)_R$ anomaly in the continuum limit, similar to the $N=2$ case discussed in section 3.1. Here, we briefly elaborate on this.

It is most convenient for the following discussion to work in terms of the six dimensional theory one gets by orbifolding with respect to $Z_r$ only. The closed string spectrum contains a twisted R-R 1-form, which we denote $U$.[21] The fractional D2 branes are magnetic sources for this field (or electric sources for its dual $\tilde{C}^{(3)}$). The D2 branes are defined as having:

$$\int_{S^2} dU = \frac{b_0}{N} \tag{3.21}$$

where $S^2$ is any two sphere in the $3,8,9$ space surrounding only one 2-brane. The factor $b_0/N$ is the R-R charge of a fractional brane, as computed in section 3.2.

The tadpole of the R-R twisted vector is determined by the $N$ sources at coordinates $\mathbf{X}_k$ (3.11), similar to (3.12):

$$(dU)^{ab} \sim \frac{b_0}{N}\, \epsilon^{abc} \partial_c \sum_{k=1}^{N} \frac{1}{|\mathbf{X}-\mathbf{X}_k|}\,, \tag{3.22}$$

in an obvious notation. Clearly, the field $U$ is the sum of the vector potentials for magnetic monopoles located at $\mathbf{X}_k$. We will not need the general expression for $U$, since near the probe brane and in the large-N limit the r.h.s. of eq. (3.22) simplifies: as we already showed in our discussion of the scalar tadpole (3.15), there $\sum_{k=1}^{N} \frac{1}{|X^a - X^a_k|} \sim \log|\rho + i\eta|$, where $\rho$ and $\eta$ are essentially $X^8$ and $X^9$ parametrizing the tangent plane to the circle. Indexing $\rho$, $\eta$ by $i,j=1,2$, we have from (3.22):

$$(dU)^{i3} \sim \epsilon^{ij3} \partial_j \log|\rho + i\eta|\,. \tag{3.23}$$

Thus, near the probe brane the expression for the RR twisted tadpole simplifies:

$$U_3 \sim \arg(\rho + i\eta)\,. \tag{3.24}$$

The Chern-Simons coupling to the twisted RR field on the probe brane world-volume $\int_{\mathcal{V}_3} U \wedge F$ [4] leads, in the classical background (3.24), to:

$$\int d^3x\, U_3\, \partial_m X^3\, F_{pq}\, \epsilon^{mpq} \sim \arg A \int d^3x\, \partial_m X^3\, F_{pq}\, \epsilon^{mpq} \tag{3.25}$$

and gives rise to the 4d anomaly (recall that $X^3$, the position of the brane in the compact direction, becomes the third component of the gauge field) exactly as in the $N=2$ case discussed in section 2.1.

[21] This field is untwisted with respect to $Z_N$, and therefore propagates in 6 dimensions. It is a combination of fields from the 3 dimensional hyper- and vector multiplets. Like all closed string fields relevant for us, it is in the $m=0$ Fourier mode with respect to the $Z_r$ action.

### 3.4 R-symmetry and the Wilson term: a geometrical picture

Issues pertaining to fermion doublers and the anomalous R-symmetry away from the continuum limit are best visualized in the string theory embedding described above. Though it is simple enough to resolve such issues in the present context purely by field theoretical means, the stringy methods are expected to help in more complex situations.

In our setup we have $N$ fractional branes which are distributed in an array in the $X^3, X^8$ plane, which is in turn embedded in the $X^3, X^8, X^9$ space. In the continuum limit the branes are distributed evenly on an approximate (for large-$N$) circle, which is to be considered as the direction conjugate (or T-dual) to the compact deconstructed direction.

Let us consider the open string fields on the fractional branes. On each brane one has 3 massless scalar fields, corresponding to fluctuations of the brane in the space spanned by $X_3, X_8, X_9$. In the continuum limit two of those are to be considered as a complex 4-dimensional scalar field, and the remaining one as a Wilson line of the 4-dimensional gauge field around the compact direction.

A priori this is a local freedom, at each location of a fractional brane. This is represented in the field theory language as a field redefinition freedom, which was used extensively in section 2. Such choices differ, away from the continuum limit, by the value of the Wilson line turned on.

We choose to identify the complex scalar field as the fluctuations in the $X^8, X^9$ direction, away from the location of a preferred brane, say the one with $k = 0$. In other words we identify $\rho, \eta$, defined above, as the complex scalar field. This corresponds to turning off the Wilson line, and as we see shortly, reproduces the Wilson R-symmetry breaking term.

The identification of open string fields corresponds to a definition of the R-symmetry transformation away from the continuum limit. In general an R-symmetry transformation will be a rotation around some axis in the $X^3, X^8, X^9$ space. Our choice of fields corresponds to choosing the axis of rotation to be tangent to the circle of branes at the point $k = 0$.

It is clear then, that the locations of the $k \neq 0$ branes away from the chosen tangent to the circle breaks the R-symmetry. Consider for definiteness the tangent at the position of the $k = 0$ brane, which, in the coordinates of (3.11), is at $X^3 = 0$, $X^8 = v$. The continuum limit R-symmetry is then an SO(2) rotation in the $X^3, X^8, X^9$ space around the axis $X^8 = v, X^9 = 0$ (this axis is shown on figure 2).

Recall now that the locations of the branes in the $X^3$–$X^8$ plane correspond to mass terms in the deconstructed theory. Having each momentum mode arise from a brane at a different location in $X^8$ corresponds exactly to the (supersymmetric completion of) Wilson R-symmetry-breaking term. The masses of the adjoint supermultiplet due to strings between the 0-th and $k$-th brane are equal to the lengths of the strings stretched between the two branes. The fermion mass itself is complex and can be written as:

$$m_k = e^{i\frac{2\pi k}{N}}\ v - v = (X^8_k - v) + i\ X^3_k\,, \tag{3.26}$$

where $X^{3(8)}_k$ are the locations (3.11) of the $k$-th brane. The utility of writing the mass this way is to separate the R-symmetry preserving (parallel to $X^3$) and R-symmetry breaking (perpendicular to $X^3$) parts. The dispersion relation that follows from (3.26) is:

$$v^{-2}\omega^2(k) = v^{-2}\,|m_k|^2 = \sin^2\frac{2\pi k}{N} + 4\;\sin^4\frac{\pi k}{N}\,. \tag{3.27}$$

Comparing (3.27) with (2.18) identifies the Wilson term coefficient $r = 1$. The value of the Wilson term coefficient is thus of order the inverse lattice spacing, is fixed by supersymmetry and is not a free parameter.


## Acknowledgments

This work is supported by the National Science and Engineering Research Council of Canada. E.P. acknowledges the hospitality and support of the "Braneworld and Supersymmetry" Workshop at the Pacific Institute for Mathematical Sciences during the initial stages of this work.


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